Transient Analysis of Systems with Exponential Transmission Lines

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Abstract—**Two computer solutions are given for pulse propagation along exponentially tapered transmission lines with arbitrary nonlinear sending and receiving networks. The solutions allow series and shunt loss terms per unit of length of the line. The method of characteristics is shown to be computationally superior to the cubic spline method in terms of accuracy and efficiency.**

Introduction

THE LITERATURE on the transient analysis of systems connected by transmission lines is rich indeed. Methods of analysis include Laplace transforms [1], finite differences [2], [3], and Bergeron's method of characteristics [4], [5]. The method of characteristics was extended to treat transmission-line problems with ground return (frequency-dependent parameters) [6], [7]. The Laplace transform method is limited to transmission lines with particular end conditions. The finite-difference solutions suffer from discretization ripple associated with replacing a distributed system by a lumped-parameter one. The method of characteristics maintains the wave character of the solutions and allows the ends to be terminated by general networks.

The exponential transmission line (ETL) has received considerable attention in the past four decades as an example of nonuniform transmission lines. A historical bibliography [8] establishes the literature available up to 1955. Applications of nonuniform and, in particular, exponential transmission lines have included impedance matching sections [9]–[11], traveling wave transformers, and resonators [12], [13]. In all of the previously cited literature, the ETL was investigated in the frequency domain. Transients in nonuniform lines have been attacked by transform methods for special end conditions [14]. A moment method has been employed to treat lines with linear taper [15].

The application that prompted this work involved an ETL as a transformer between a nonlinear sending network and a linear receiving network. The transient response of this system was required. To treat this problem, two methods of analysis were considered: cubic spline representation [16] and Bergeron's characteristics [17].

The transient-response problem is solved by the two approaches (splines and characteristics). The spline method suffers from the same discretization ripple found in finite-difference solutions and is less computationally efficient than the method of characteristics. The method of characteristics does not produce the troublesome ripple and it maintains the rise time of the propagating pulse.

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The Dynamic System

The dynamic system of interest consists of a sending network and a receiving network connected by an ETL. The system is sketched in Fig. 1. $\boldsymbol{S}(t_1), \boldsymbol{R}(t_1)$ are the state vectors of the sending and receiving networks. The functions $\boldsymbol{F}_s(t_1,\boldsymbol{S})$ and $\boldsymbol{F}_r(t_1,\boldsymbol{R})$ are vector-valued functions which define the derivatives of $\boldsymbol{S}(t_1)$ and $\boldsymbol{R}(t_1)$. The last member of the state vector $\boldsymbol{S}(t_1)$ must be either the input voltage $V(0,t_1)$ or current $I(0,t_1)$ of the ETL. Likewise, the output voltage $V(l,t_1)$ or current $I(l,t_1)$ must be the last member of the state vector $\boldsymbol{R}(t_1)$. The functions $\boldsymbol{F}_s(t_1,\boldsymbol{S})$ and $\boldsymbol{F}_r(t_1,\boldsymbol{R})$ may be nonlinear.

Governing Equations of the System

The sending and receiving networks are governed by sets of nonlinear first-order differential equations of the following form:

$$\dot{\boldsymbol{S}} = \boldsymbol{F}_s(t_1,\boldsymbol{S}) \tag{1}$$

and

$$\dot{\boldsymbol{R}} = \boldsymbol{F}_r(t_1,\boldsymbol{R}) \tag{2}$$

where the functions $\boldsymbol{F}_s(t_1,\boldsymbol{S})$ and $\boldsymbol{F}_r(t_1,\boldsymbol{R})$ are sufficiently smooth to insure the existence of unique solutions to the initial value problem.

The ETL is governed by the standard transmission-line equations as given by Johnson [18]:

$$\frac{\partial V}{\partial x_1} + L\frac{\partial I}{\partial t_1} + RI = 0 \tag{3}$$

$$C\frac{\partial V}{\partial t_1} + GV + \frac{\partial I}{\partial x_1} = 0 \tag{4}$$

where $V(x_1,t_1)$ and $I(x_1,t_1)$ have been previously defined, and $C(x_1)$, $L(x_1)$, $R(x_1)$, and $G(x_1)$ are the shunt capacitance, series inductance, series resistance, and shunt conductance per unit of length of the line, respectively.

For an ETL, the electrical properties vary as follows:

$$C(x_1) = C_0 e^{-\alpha x_1}$$

$$L(x_1) = L_0 e^{\alpha x_1}$$

$$R(x_1) \quad \text{and} \quad G(x_1) \text{ are arbitrary} \tag{5}$$

where C_0 and L_0 are the values of the capacitance and inductance at $x_1 = 0$. The parameter α defines the taper of the ETL.

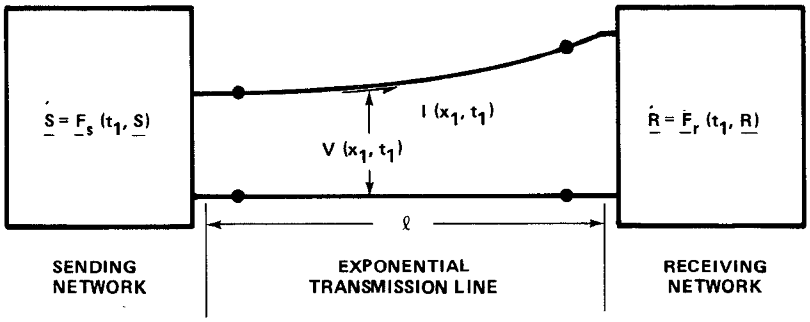

Fig. 1. The dynamic system.

For convenience, dimensionless spatial and temporal coordinates are introduced into (3)–(5) by the transformation

$$x_1 = lx \qquad t_1 = \sqrt{L_0 C_0}\, lt. \tag{6}$$

Equation (5) becomes

$$\begin{aligned} C(x) &= C_0 e^{-2\beta x} \\ L(x) &= L_0 e^{2\beta x} \\ R(x) &= lR(x_1) \\ G(x) &= lG(x_1) \end{aligned} \tag{7}$$

where

$$\beta = \alpha l/2.$$

Equations (3) and (4) become

$$\frac{\partial V}{\partial x} + Z(x)\frac{\partial I}{\partial t} + R(x)I = 0 \tag{8}$$

$$\frac{\partial V}{\partial t} + G(x)Z(x)V + Z(x)\frac{\partial I}{\partial x} = 0 \tag{9}$$

where

$$Z(x) = \sqrt{\frac{L_0}{C_0}}\, e^{2\beta x} = Z_0 e^{2\beta x}.$$

Because the speed of propagation of a weak[1] signal along a transmission line is given by

$$c_p = [L(x_1)C(x_1)]^{-1/2} = (L_0 C_0)^{-1/2}$$

it is constant for an ETL. The unit of the dimensionless time t corresponds to the time required for a signal to propagate a distance l along the ETL.

The calculation of the dynamic response of the ETL with nonlinear terminal networks requires the solution of the sets of ordinary differential equations (1) and (2) and partial differential equations (8) and (9). Numerically, the set of ordinary differential equations presents no difficulties; the equations can be solved by any standard method of numerical integration. The partial differential equations are treated here by two distinctly different methods: 1) cubic spline representation of the spatial variation and 2) the method of characteristics. The cubic spline representation reduces the partial differential equations to sets of linear ordinary first-order differential equations. The method of characteristics involves integrating the partial differential equations along their characteristics. These two approaches are presented in the next two sections.

[1] Continuous across the wavefront but has a discontinuity in the first derivative.

Cubic Spline Representation

The voltage $V(x,t)$ and current $I(x,t)$ are discretized by their values at a set of nodes along the ETL. The functions $V(x,t)$ and $I(x,t)$ are assumed to vary as cubic spline functions $\phi_j(x)$ defined for the set of nodes x_k on the interval $0 \le x \le 1$ as described by Ahlberg *et al.* [16]. The functions $\phi_j(x)$ are sectionally cubic and have continuous derivatives through the second order over the interval $0 < x < 1$. The spline functions $\phi_j(x)$ take on the values of the Kronecker delta

$$\phi_j(x_k) = \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \ne k \end{cases} \qquad \text{for} \quad j, k = 1, 2, \cdots, n$$

at the nodes and all have vanishing second-order derivatives at $x = 0$ and $x = 1$. This will allow the functions $V(x,t)$ and $I(x,t)$ to have nonzero values and slopes at the ends of the ETL.

The functions $V(x,t)$ and $I(x,t)$ are approximated as

$$V(x,t) = \sum_{j=1}^{n} \phi_j(x)V_j(t) \tag{10}$$

$$I(x,t) = \sum_{j=1}^{n} \phi_j(x)I_j(t). \tag{11}$$

The unknowns $V_j(t)$ and $I_j(t)$ are the physical values of $V(x,t)$ and $I(x,t)$ at the node x_j. Substituting (10) and (11) into (8) and (9) yields

$$\sum_{j=1}^{n} [\phi_j'(x)V_j + Z(x)\phi_j(x)\dot{I}_j + R(x)\phi_j(x)I_j] = e_1(x,t) \tag{12}$$

$$\sum_{j=1}^{n} [\phi_j(x)\dot{V}_j + G(x)Z(x)\phi_j(x)V_j + Z(x)\phi_j'(x)I_j] = e_2(x,t). \tag{13}$$

Equations (12) and (13) are not identically satisfied as indicated by the error terms $e_1(x,t)$ and $e_2(x,t)$. Obtaining a determinate set of equations requires that the errors $e_1(x,t)$ and $e_2(x,t)$ vanish at the nodes x_i, $i = 1, 2, \cdots, n$. This yields

$$\sum_{j=1}^{n} \phi_j'(x_i)V_j + Z(x_i)\dot{I}_i + R(x_i)I_i = 0 \tag{14}$$

$$\dot{V}_i + G(x_i)Z(x_i)V_i + \sum_{j=1}^{n} Z(x_i)\phi_j'(x_i)I_j = 0, \quad i = 1, 2, \cdots, n. \tag{15}$$

Equations (14) and (15) represent a set of first-order linear differential equations for the determination of $V_j(t)$ and $I_j(t)$.

Equations (1), (2), (14), and (15) now represent the total system. There is a certain redundancy in this collective set of equations. Because either $V_1(t)$ or $I_1(t)$ is a member of the state vector $\boldsymbol{S}(t)$ of the sending network, equation (1) furnishes an expression for either $\dot{V}_1$ or $\dot{I}_1$, depending upon which is included in $\boldsymbol{S}$. If $V_1(t)$ is the last member of $\boldsymbol{S}$, the last of (1) for $\dot{V}_1$ is used and the first of (14) is discarded. On the other hand, if I_1 is a member of $\boldsymbol{S}$, the first of (15) is discarded. Likewise, either the last of (14) or (15) is discarded if $V_n(t)$ or $I_n(t)$ is the last member of $\boldsymbol{R}$. The equations of sets (1) or (2) are retained because these differential equations form boundary conditions on the partial differential equations of the ETL represented by (14) and (15). The collective set of differential equations as modified are numerically integrated with suitable initial conditions to obtain the dynamic response of the system.

Method of Characteristics

Partial differential equations (8) and (9) which govern the ETL are of the hyperbolic type [19]. Hyperbolic partial differential equations have the property that they can propagate discontinuities along certain or characteristic lines in their solution space. The solution space is a semi-infinite strip in the x,t plane defined by

$$0 < x < 1 \qquad 0 < t < \infty.$$

The characteristic lines of the ETL are $x + t =$ constant and $x - t =$ constant. The characteristic lines suggest new coordinates as

$$\xi = t + x \qquad \eta = t - x \tag{16}$$

so that $\xi =$ constant and $\eta =$ constant are the characteristics of the partial differential equations. Transforming (8) and (9) to ξ,η coordinates by (16) then adding and subtracting yields

$$\frac{\partial V}{\partial \xi} + Z(\xi,\eta)\frac{\partial I}{\partial \xi} + \frac{1}{2}[R(\xi,\eta)I + G(\xi,\eta)Z(\xi,\eta)V] = 0 \tag{17}$$

$$\frac{\partial V}{\partial \eta} - Z(\xi,\eta)\frac{\partial I}{\partial \eta} + \frac{1}{2}[G(\xi,\eta)Z(\xi,\eta)V - R(\xi,\eta)I] = 0. \tag{18}$$

This method is based upon the form of (17) and (18). In particular, the presence of partial derivatives with respect to ξ only in (17) and with respect to η only in (18) allows them to be integrated approximately along constant η lines and constant ξ lines, respectively. This process will yield algebraic equations for the calculation of V and I at a point in the solution space x,t. Only the special case of zero initial values of $V(x,t)$ and $I(x,t)$ and constant values of $R(x)$ and $G(x)$ as R_0 and G_0 along the ETL will be considered.

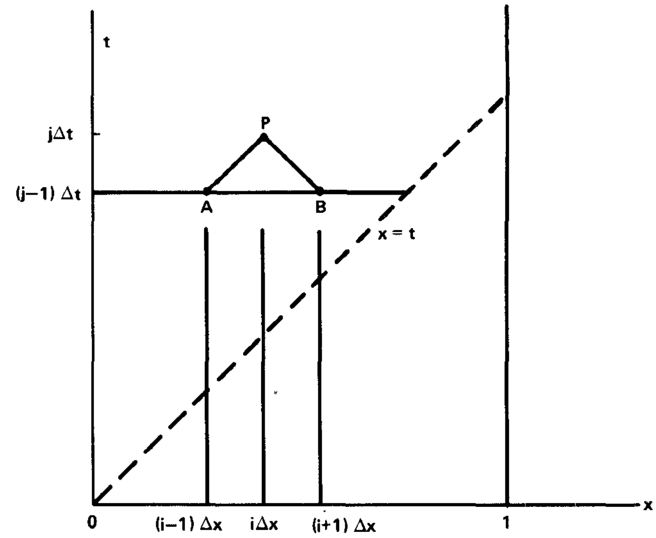

Fig. 2. Solution space for an ETL.

Because the waveform travels down the line with a velocity of one (in this set of dimensionless coordinates), both $V(x,t)$ and $I(x,t)$ are zero until $t > x$. To develop the numerical method of treating the ETL, the line is first divided into a number, n, of equal-length intervals. The solution is calculated at time 0, Δt, $2\Delta t$, $\cdots$, where Δt is numerically equal to the length of the intervals along the line, because the electrical signal travels one step along the line in a period of time

$$\Delta t = 1/n.$$

Assuming that the solution is known at $t = (j-1)\Delta t$, calculate V and I along the line at $t = j\Delta t$. The solution will be evaluated at the stations $x = i\Delta x$ along the line and will be designated as

$$V(i\Delta x, j\Delta t) = V_i^j$$

$$I(i\Delta x, j\Delta t) = I_i^j$$

if $i \geq j$,

$$V_i^j = I_i^j = 0.$$

For $0 < i < n$ and $j > i$, the solution space sketched in Fig. 2 is considered. Equation (17) is integrated along the η line from A to P

$$V_i^j - V_{i-1}^{j-1} + \int_{\xi_A}^{\xi_P} Z_0 e^{\beta(\xi-\eta)} \frac{\partial I}{\partial \xi} d\xi + \tfrac{1}{2}\int_{\xi_A}^{\xi_P} [R_0 I + G_0 Z_0 e^{\beta(\xi-\eta)} V] \, d\xi = 0 \tag{19}$$

where

$$\xi_A = (i-1)\Delta x + (j-1)\Delta t$$

$$\xi_P = i\Delta x + j\Delta t$$

$$\eta = j\Delta t - i\Delta x.$$

Equation (18) is integrated along the ξ line from B to P

$$V_i^j - V_{i+1}^{j-1} - \int_{\eta_B}^{\eta_P} Z_0 e^{\beta(\xi-\eta)} \frac{\partial I}{\partial \eta} d\eta + \tfrac{1}{2}\int_{\eta_B}^{\eta_P} [G_0 Z_0 e^{\beta(\xi-\eta)} V - R_0 I]\, d\eta = 0 \quad (20)$$

where

$$\eta_B = (j-1)\Delta t - (i+1)\Delta x$$
$$\eta_P = j\Delta t - i\Delta x$$
$$\xi = i\Delta x + j\Delta t.$$

The trapezoidal rule

$$\int_{x_A}^{x_B} f(x)\, dx \doteq \tfrac{1}{2}[f(x_B) + f(x_A)](x_B - x_A)$$

is used to evaluate the integrals so that (19) and (20) can be written as

$$A_{11} V_i^j + A_{12} I_i^j = b_1$$
$$A_{21} V_i^j + A_{22} I_i^j = b_2 \quad (21)$$

where

$$A_{11} = 1 + \frac{G_0 Z_0}{2} \Delta x e^{2i\beta\Delta x}$$
$$A_{12} = \tfrac{1}{2}[R_0 \Delta x + Z_0 e^{2\beta i \Delta x}(1 + e^{-2\beta\Delta x})]$$
$$A_{21} = A_{12}$$
$$A_{22} = -\tfrac{1}{2}[R_0 \Delta x + Z_0 e^{2\beta i\Delta x}(1 + e^{-2\beta\Delta x})]$$
$$b_1 = \left[1 - \frac{G_0 Z_0}{2}\Delta x e^{2\beta(i-1)\Delta x}\right] V_{i-1}^{j-1} + \tfrac{1}{2}[Z_0 e^{2\beta i\Delta x}(1 + e^{-2\beta\Delta x}) - R_0 \Delta x] I_{i-1}^{j-1}$$
$$b_2 = \left[1 - \frac{G_0 Z_0}{2}\Delta x e^{2\beta(i+1)\Delta x}\right] V_{i+1}^{j-1} - \tfrac{1}{2}[Z_0 e^{2\beta i\Delta x}(1 + e^{2\beta\Delta x}) - R_0 \Delta x] I_{i+1}^{j-1}.$$

Equation (21) holds for $j \geq 1$, $1 \leq i \leq n-1$ with $j > i$. Under these conditions, equation (21) can be solved for V_i^j and I_i^j in terms of V_{i-1}^{j-1}, V_{i+1}^{j-1}, I_{i-1}^{j-1}, and I_{i+1}^{j-1}. Thus the two new values at $x = i\Delta x$ can be evaluated in terms of the voltage and current at $t = (j-1)\Delta t$ at the positions $x = (i-1)\Delta x$ and $x = (i+1)\Delta x$. This limited domain of dependence of the solution of the transmission-line equations is another property of hyperbolic partial differential equations. Equation (21) cannot determine the solution at the sending end and the receiving end of the ETL without considering the nonlinear differential equations (1) and (2) which govern the terminal networks.

At the sending end of the ETL the first equation of (21) is not valid because no η line goes from the point $x = 0$ back into the solution space of the ETL. Thus this equation must be discarded at $x = 0$. The second equation of (21) together with (1) of the sending network can be integrated to obtain V_i^j and I_i^j. Similarly at $x = 1$ the second equation of (21) must be discarded; equation (2) along with the first equation of (21) furnishes the needed information to calculate V_n^j and I_n^j.

Numerical Results and Conclusions

The two distinct methods of analysis based upon cubic spline representation and characteristics were used to calculate the response of a particular exponentially tapered transmission line with a nonlinear sending network and with the receiving network as just a resistive load. The sending network was a simple transistor circuit. The state vector $\boldsymbol{S}(t)$ contained the two loop currents and two transistor model parameters associated with the equivalent circuit. The current in the second loop of the sending network is the input current to the ETL.

Digital computer programs were written for the two approaches to the system response. In both programs, the numerical integration was done by a fourth-order Runge–Kutta subroutine. Because the primary concern of this work is the analysis of the ETL, the sending network will not be described in detail. The sending network generates a signal that is input to the ETL; the waveform is transmitted down the line; and the network at $x = 1$ receives the output signal. Of interest here is the evaluation of the ability of these two methods to calculate numerically the solution of the partial differential equations which govern the ETL. For this reason the numerical results will be limited to a comparison of the output voltage with the input voltage and a presentation of the voltage waveform as it travels down the transmission line.

The output voltage and input voltage calculated by the cubic spline representation are presented in Fig. 3. The signals are not smooth and appear to be the sum of the true response and some noise which is a result of the numerical process used to calculate them. The rise time of the output voltage is approximately twice as long as the rise time of the input signal. The voltages are normalized so that the maximum input voltage is unity. The maximum output voltage is 7.79.

Fig. 4 presents the voltage waveform as it travels down the ETL. The noise described is evident in these signals. The vertical dashed lines mark the position to which the signal should have propagated in the time indicated. The fact that the signals are not zero at the marked positions indicates the presence of "forerunners." Because the signals cannot physically travel past the marked positions, these forerunners are evidence of calculation errors.

Similar numerical results were calculated by the method of characteristics. Fig. 5 gives the comparison of the output and input voltage of the ETL obtained by this method. The responses are very smooth. The rise time of the output signal is the same as that of the input signal. The maximum voltage obtained is 7.50. Fig. 6 depicts the voltage waveform as it travels down the transmission line. The calculation method assures the absence of forerunners.

Comparing the numerical results obtained by the two methods of treating the ETL with terminal networks leads to the following conclusions.

1) The method of characteristics maintains the rise time

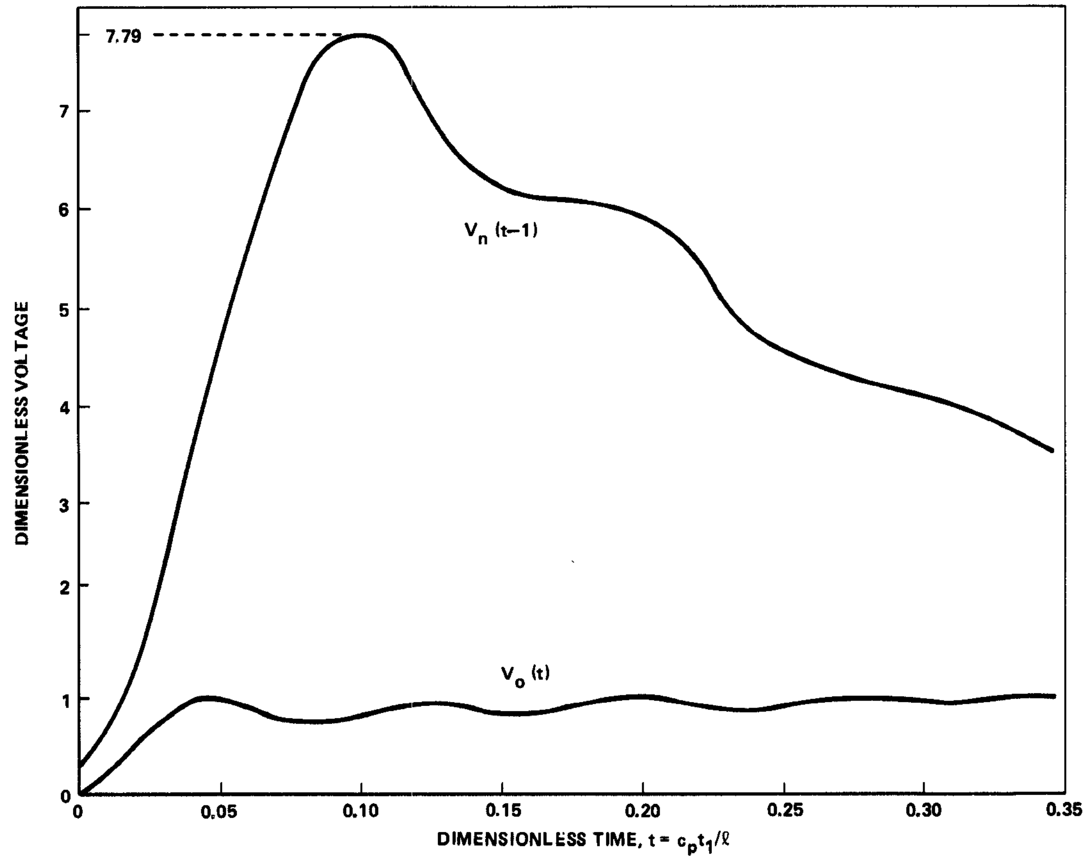

Fig. 3. Comparison of output voltage and input voltage calculated by cubic spline representation.

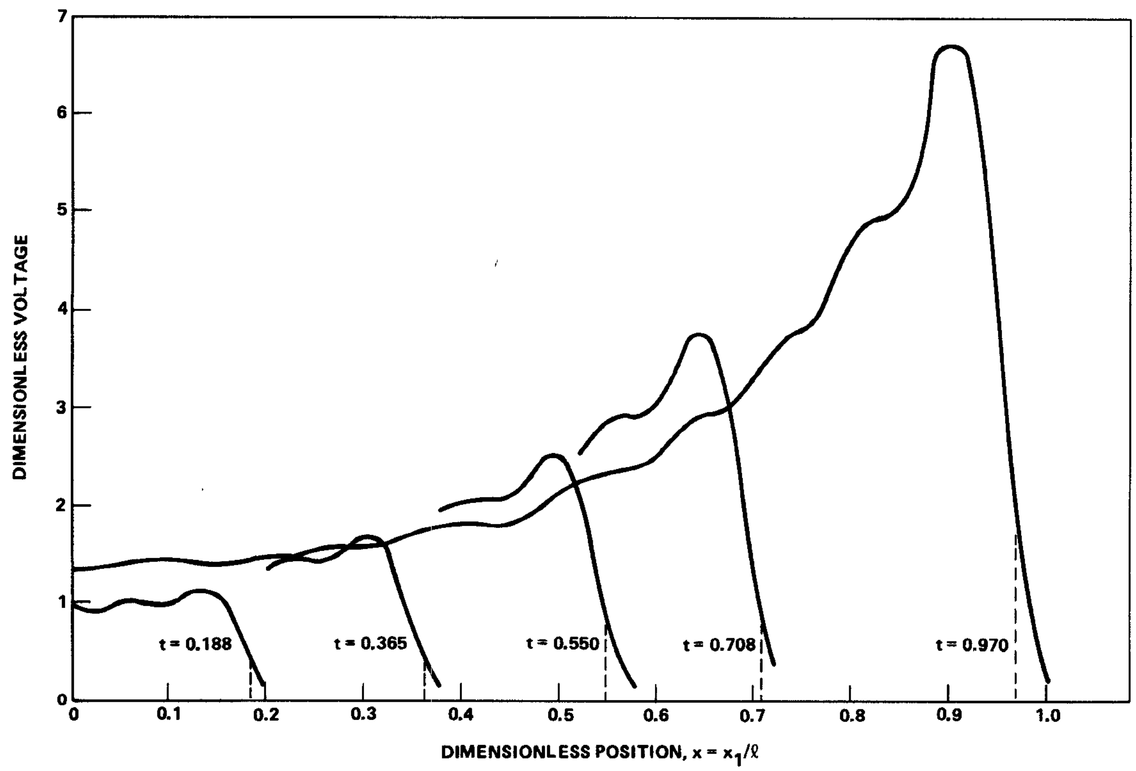

Fig. 4. Voltage waveforms calculated by cubic spline representation.

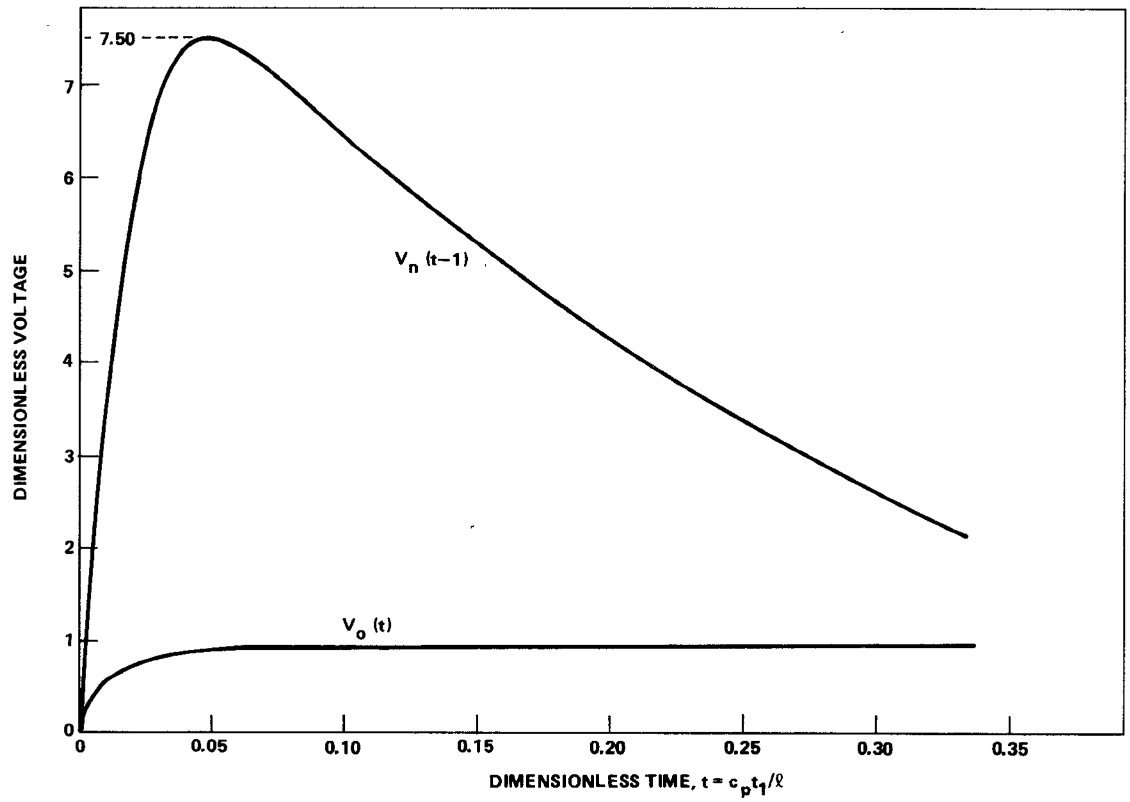

Fig. 5. Comparison of output voltage and input voltage calculated by the method of characteristics.

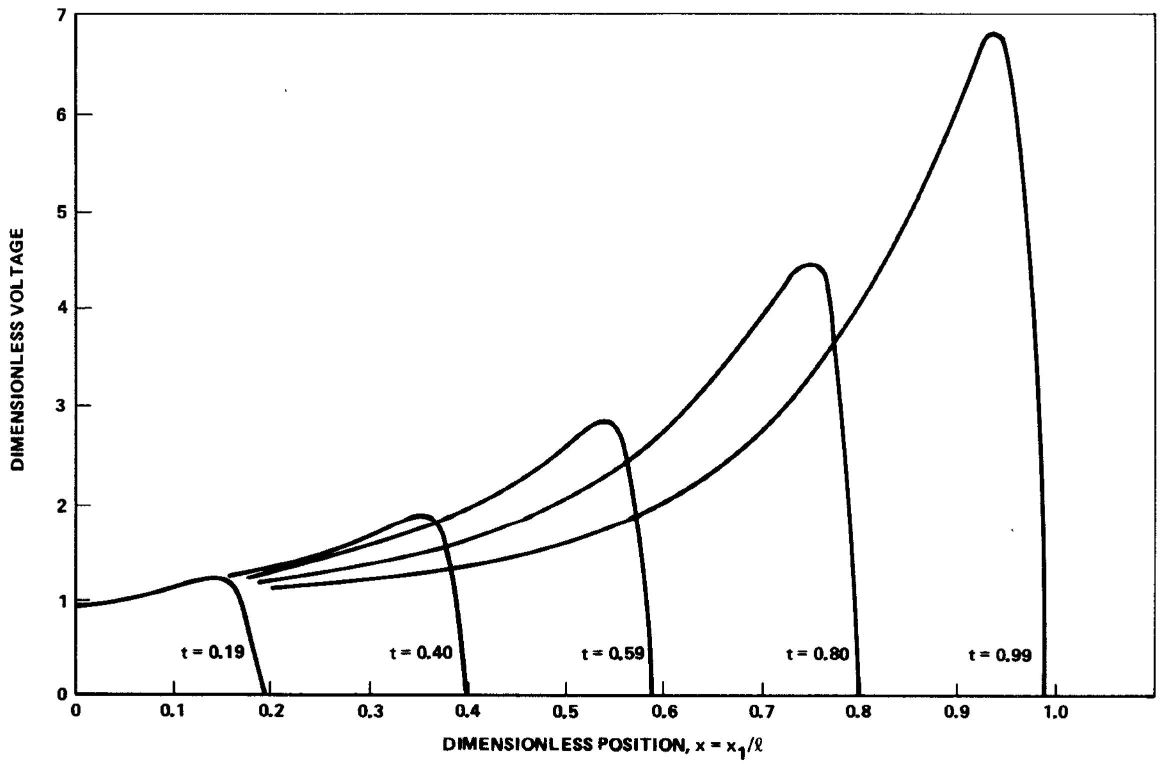

Fig. 6. Voltage waveforms calculated by the method of characteristics.

of the signal as it travels the length of the transmission line. The cubic spline method does not.

2) The cubic spline method produces discretization ripple in the response. This discretizing error is similar to that caused by replacing continuous systems by a lumped-parameter discrete system.

In addition to comparing the numerical results, some comments are in order with regard to the efficiency and storage requirements of the two methods.

1) The cubic spline method requires the integration of two first-order differential equations at each node along the transmission line. The system treated had four variables in the sending network with 51 nodes for a total of 106 first-order differential equations to integrate. With 51 nodes, the program required in excess of 25 000 storage locations. The integration of the response required over 5 min of computation time on the CDC 6600 computer.

2) Using the method of characteristics, the voltage and current were calculated at 501 stations along the transmission line. To calculate the voltage and current at a given position for the new value of time requires the solution of two simultaneous algebraic equations. The program required approximately 10 000 storage locations. A typical run required less than 60 s of computation time.

From the preceding comparisons, it is clear that the method of characteristics is the superior of these two methods of computation.

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